# Mathematics 1 AESB1110-15: Test 1 - ANSWERS

October 30, 2016

**Rules:**

- No points are assigned for a question if only the final answer is given without any intermediate steps (except Question 4).
- Subtract 0.25 p. for the first occurrence of an arithmetic error. Do not subtract further points if the same error 'propagates' into subsequent calculations.

**Question 1:** Simplify: $\sin(\tan^{-1}(x))$ $\boxed{2\,p.}$

**Answer:**

$$\tan^{-1}(x) = y \quad \Leftrightarrow \quad \tan(y) = x \quad \Rightarrow \quad \sin(\tan^{-1}(x)) = \sin(y) = ?; \quad \boxed{+1\,p.}$$

$$\tan(y) = \frac{\sin(y)}{\cos(y)} = x \quad \text{and} \quad \sin^2(y) + \cos^2(y) = 1;$$

$$\sin(y) = x\cos(y) \quad \text{and} \quad x^2\cos^2(y) + \cos^2(y) = 1 \quad \Rightarrow \quad (x^2+1)\cos^2(y) = 1;$$

$$\sin(y) = x\cos(y) \quad \text{and} \quad \cos^2(y) = \frac{1}{1+x^2} \quad \Rightarrow \quad \cos(y) = \frac{1}{\sqrt{1+x^2}};$$

$$\sin(\tan^{-1}(x)) = \sin(y) = x\cos(y) = \frac{x}{\sqrt{1+x^2}} \quad \boxed{+1\,p.}$$

Give $\boxed{1.5\,p.}$ if the problem is solved geometrically.

**Question 2:** Given the function $\boxed{2\,p.}$

$$f(x) = \begin{cases} \frac{x^4-1}{1-x} & \text{if} \quad x \neq 1; \\ a & \text{if} \quad x = 1; \end{cases}$$

for what value of $a$ is $f(x)$ continuous at $x = 1$?

**Answer:** $f(x)$ is continuous at $x = 1$ iff $\lim_{x\to 1} f(x) = f(1)$ $\boxed{+0.5\,p.}$

$f(1) = a$; $\boxed{+0.5\,p.}$; $f(x)$ is the same for $x \to 1^-$ and $x \to 1^+$, so:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^4-1}{1-x} = \left[\text{e.g. l'Hospital, since } \frac{0}{0}\right] = \lim_{x\to 1} \frac{4x^3}{-1} = -4, \quad \boxed{+0.5\,p.}$$

Computing $x \to 1^-$ and $x \to 1^+$ limits separately is also OK

$\lim_{x\to 1} f(x) = f(1)$, i.e., $f(x)$ is continuous at $x = 1$, iff $a = -4$. $\boxed{+0.5\,p.}$

**Question 3:** Use (a) the definition of inverse function and (b) implicit differentiation to prove that 2 p.

$$[\cos^{-1}(x)]' = -\frac{1}{\sqrt{1-x^2}}$$

**Answer:**

(a) $\cos^{-1}(x) = y \quad \Leftrightarrow \quad x = \cos(y);$ $\boxed{+0.75\ p.}$

(b) $(x)' = [\cos(y)]' \quad \Rightarrow \quad 1 = -\sin(y)\,y' \quad \Rightarrow$ $\boxed{+0.75\ p.}$

$$y' = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}} \quad \boxed{+0.5\ p.}$$

**Question 4:** Differentiate the following function: 2 p.

$$y(x) = \frac{1}{\cos^{-1}(x)}$$

**Answer:**

$$y' = -\frac{1}{[\cos^{-1}(x)]^2}\,[\cos^{-1}(x)]' \quad \boxed{+1\ p.}$$

$$= \frac{1}{[\cos^{-1}(x)]^2\sqrt{1-x^2}} \quad \boxed{+1\ p.}$$

**Question 5:** Use linear approximation to estimate $1.001^{100}$ 2 p.

**Answer:**

$f(x) \approx L(x) = f(x_0) + f'(x_0)\,(x - x_0)$ $\boxed{+0.5\ p.}$

$f(x) = x^{100}; \quad x = 1.001; \quad x_0 = 1$ $\boxed{+0.5\ p.}$

$f(x_0) = f(1) = 1^{100} = 1; \quad f'(x) = 100x^{99}; \quad f'(x_0) = f'(1) = 100;$ $\boxed{+0.5\ p.}$

$1.001^{100} \approx 1 + 100(1.001 - 1) = 1 + 0.1 = 1.1$ $\boxed{+0.5\ p.}$